CMB two- and three-point correlation functions from Alfvén waves

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We study the cosmic microwave background (CMB) temperature fluctuations non-gaussianity due to the vector mode perturbations (Alfvén waves) supported by a stochastic cosmological magnetic field. We present detailed derivations of the statistical properties, two and three point correlation functions of the vorticity perturbations and corresponding CMB temperature fluctuations.

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I. INTRODUCTION

In the framework of the standard cosmological scenario the cosmic microwave background (CMB) temperature fluctuations are gaussian and are fully determined by the CMB temperature fluctuation two-point correlation functions while higher order odd correlation functions (for example three-point correlations) are identically zero (for a review on CMB fluctuations and possible non-gaussianity see Ref. [1] and references therein). This is a consequence of the inflationary scenario that predicts the gaussian initial perturbations, and even at the level of perturbations there is no violation of rotational symmetry[1]. Several observations indicate that the CMB temperature map could be slightly non-gaussian [3] and thus to adequately describe the CMB fluctuations one must study higher order correlations functions. Furthermore, some modifications of standard inflationary models lead to a slightly non-gaussian CMB map [4].

A common way to characterize the CMB temperature fluctuations non-gaussianity is to introduce the $f_{\rm NL}$ parameter, which in fact determines the relation between the two-point and three-point correlation functions [5]. The current CMB data limits $f_{\rm NL} = 32 \pm 21$ [3]. PLANCK mission will be able to give us with stronger limits (to improve current limit by an order or a few). If the nearest future CMB measurements, for example PLANCK data will confirm that $|f_{\rm NL}|$ is not significantly less then one, the standard cosmological scenario must be revised substantially. There are several different ways for such a revision. For example the inflation could be driven by multiple fields, or the Universe isotropy has been violated at very early epochs [1]. Recent studies [6, 7] address the magnetic field induced density perturbations as a source of the CMB temperature fluctuations non-gaussianity. The physical meaning of this effect is as follows: the temperature anisotropies caused by the magnetic field are proportional to the magnetic field energy density parameter, $\Delta T/T \propto B^2/\rho_{\rm cr}$ [6], where $\rho_{\rm cr}$ is the critical density today and B is the comoving value of an effective magnetic field. Accounting that the square of the magnetic field B^2 is a non-linear form [8], the corresponding $\Delta T/T$ fluctuations must be non-gaussian. The limits for the scale invariant magnetic field amplitude from the CMB non-gaussianity test is of order of 10^{-9} Gauss [9].

In this paper we investigate the CMB non-gaussianity due to the vector mode of perturbations induced by a stochastic magnetic magnetic field, (see Refs. [10–12] for details of the vector magnetized mode). If the magnetic field presence is a reason for the CMB non-gaussianity, this magnetic field must satisfy several conditions: (i) the magnetic field must be generated in the early Universe, prior to recombination; There are different mechanisms to generate magnetic fields in the early Universe, such as inflation, phase transitions, see for reviews [13]; (ii) the correlation length of the cosmological magnetic field must satisfy the requirement of causality [14], and thus to have a field correlated over the horizon scale or even larger this field should be generated during the inflation and have a scale invariant spectrum [15]; (iii) the amplitude of this magnetic field should be small enough to preserve the isotropy of

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[1] The generation of the vector mode, which involves a preferred direction, is not excluded during the inflation, but the exponential expansion washes out the vector (vorticity) perturbations if no supporting external source is present [2].

the background Friedman-Lemaître-Robertson-Walker (FLRW) metric (so the energy density of the magnetic field should be the first order of perturbations), be below the upper bounds (few nGauss) given by observations [16]. On the other hand the amplitude of the magnetic field should be large enough to leave observational traces on CMB. Recently it was argued that non-observation of blazars in TeV range by Fermi mission indicates the presence of large scale correlated intergalactic magnetic field with a lower bound of order of 10^{-16} Gauss [17]. The existence of the lower bound of order of 10^{-16} Gauss magnetic field favors the magnetic field of cosmological origin [18], and thus the magnetic field amplitude at 1 Mpc is squeezed between 10^{-9} and 10^{-16} Gauss.

Vector and tensor modes of magnetized perturbations are much more complicated then the scalar one. The first paper to address the CMB bispectrum induced by the vector and tensor mode of perturbations has been Ref. [19] where the analytical expressions were derived. A natural extension of that approach, namely numerical estimation of the vector and tensor modes induced non-gaussianity were presented in Refs. [19–22]. In our study we give detailed derivations of the two and three point correlation functions of the CMB temperature fluctuations induced by the magnetized vector mode. In this sense at this stage this article has a methodological nature.

The outline of the rest of the paper is as follows: In Sec. II we define the magnetized vector mode of perturbations and compute the Lorentz force two- (Sec. IIA) and three- (Sec. IIB) point correlation functions. We explicitly discuss in details the difference of the vorticity perturbations two-point correlations functions, and show that the non-gaussianity of the vector field (vorticity) can be seen already from the two-point correlation function. In Sec. III we address the CMB temperature fluctuations induced by the magnetized vector mode perturbations. We present analytical expressions for the two- (Sec. IIIA) and three- (Sec. IIIB) point correlations of the CMB temperature anisotropies. We briefly discuss our results and conclude in Sec. IV. Useful mathematical formulae and details of computations are given in Appendix.

II. MAGNETIZED PERTURBATIONS VECTOR MODE

To study the dynamics of linear magnetic vector perturbations about a spatially-flat FLRW homogeneous cosmological spacetime background (described by the metric tensor $\bar{g}_{\mu\nu} = a^2\eta_{\mu\nu}$, where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric tensor and $a(\eta)$ the scale factor) we follow the standard procedure and decompose the metric tensor into a spatially homogeneous background part ($\bar{g}_{\mu\nu}$) and a perturbation part, $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$, where $\mu, \nu \in (0, 1, 2, 3)$ are spacetime indices. Vector perturbations $\delta g_{\mu\nu}$ can be described by two three-dimensional divergence-free vector fields $\mathbf{A}$ and $\mathbf{H}$ [2], where

$$\delta g_{0i} = \delta g_{i0} = a^2 A_i, \qquad\qquad \delta g_{ij} = a^2(H_{i,j} + H_{j,i}). \tag{1}$$

Here a comma denotes the usual spatial derivative, $i, j \in (1, 2, 3)$ are spatial indices, and $\mathbf{A}$ and $\mathbf{H}$ vanish at spatial infinity. Studying the behavior of these variables under infinitesimal general coordinate transformations one finds that $\mathbf{V} = \mathbf{A} - \dot{\mathbf{H}}$ is gauge-invariant (the overdot represents a derivative with respect to conformal time). $\mathbf{V}$ is a vector perturbation of the extrinsic curvature [23]. Exploiting the gauge freedom we choose $\mathbf{H}$ to be constant in time. Then the vector metric perturbation may be described in terms of two divergenceless three-dimensional gauge-invariant vector fields, the vector potential $\mathbf{V}$ and a vector representing the transverse peculiar velocity of the plasma, the vorticity $\mathbf{\Omega} = \mathbf{v} - \mathbf{V}$, where $\mathbf{v}$ is the spatial part of the four-velocity perturbation of a stationary fluid element [11].[2] As we noted in the Introduction, in the absence of a source the vector perturbation $\mathbf{V}$ decays with time (this follows from the equation for $\dot{\mathbf{V}}$, $\dot{\mathbf{V}} + 2(\dot{a}/a)\mathbf{V} = 0$) and therefore can be ignored.

The residual ionization of the primordial plasma is large enough to ensure that magnetic field lines are frozen into the plasma. Neglecting fluid back-reaction onto the magnetic field, the spatial and temporal dependence of the field separates, $\mathbf{B}(t, \mathbf{x}) = \mathbf{B}(\mathbf{x})/a^2$ [25]. Since the fluid velocity is small the displacement current in Ampère's law may be neglected; this implies the current $\mathbf{J}$ is determined by the magnetic field via $\mathbf{J} = \nabla \times \mathbf{B}/(4\pi)$. Accounting for a frozen-in magnetic field lines the induction law takes the form $\dot{\mathbf{B}} = \nabla \times (\mathbf{v} \times \mathbf{B})$. As a result the baryon Euler equation for $\mathbf{v}$ has the Lorentz force $\mathbf{L}(\mathbf{x}) = -\mathbf{B}(\mathbf{x}) \times [\nabla \times \mathbf{B}(\mathbf{x})]/(4\pi)$ as a source term. The photons are neutral so the photon Euler equation does not have a Lorentz force source term. The Euler equations for photons and baryons are [11, 12]

$$\dot{\mathbf{\Omega}}_\gamma + \dot{\tau}(\mathbf{v}_\gamma - \mathbf{v}_b) \;=\; 0, \tag{2}$$

[2] Given the general coordinate transformation properties of the velocity field $\mathbf{v}$, two gauge-invariant quantities can be constructed, the shear $\mathbf{s} = \mathbf{v} - \dot{\mathbf{H}}$ and the vorticity $\mathbf{\Omega} = \mathbf{v} - \mathbf{A}$ [23]. In the gauge $\dot{\mathbf{H}} = 0$ (i.e., $\mathbf{V} = \mathbf{A}$) we get $\mathbf{\Omega} = \mathbf{v} - \mathbf{V}$ [24].

$$\dot{\mathbf{\Omega}}_b + \frac{\dot{a}}{a}\mathbf{\Omega}_b - \frac{\dot{\tau}}{R}(\mathbf{v}_\gamma - \mathbf{v}_b) \;=\; \frac{\mathbf{L}^{(V)}(\mathbf{x})}{a^4(\rho_b + p_b)} \;, \tag{3}$$

where the subscripts γ and b refer to the photon and baryon fluids, and ρ and p are energy density and pressure. Here $\dot{\tau} = n_e \sigma_T a$ is the differential optical depth, n_e is the free electron density, σ_T is the Thomson cross section, $R = (\rho_b + p_b)/(\rho_\gamma + p_\gamma) \simeq 3\rho_b/4\rho_\gamma$ is the momentum density ratio between baryons and photons, and $L_i^{(V)}$ is the transverse vector (divergenceless) part of the Lorentz force. In the tight-coupling limit $\mathbf{v}_\gamma \simeq \mathbf{v}_b$, so we introduce the photon-baryon fluid divergenceless vorticity $\mathbf{\Omega}$ $(= \mathbf{\Omega}_\gamma = \mathbf{\Omega}_b)$ that satisfies

$$(1+R)\dot{\mathbf{\Omega}} + R\frac{\dot{a}}{a}\mathbf{\Omega} = \frac{\mathbf{L}^{(V)}(\mathbf{x})}{a^4(\rho_\gamma + p_\gamma)}. \tag{4}$$

The average Lorentz force $\langle \mathbf{L}(\mathbf{x}) \rangle = -\langle \mathbf{B} \times [\nabla \times \mathbf{B}] \rangle/(4\pi)$ vanishes, while the r.m.s. Lorentz force $\langle \mathbf{L}(\mathbf{x}) \cdot \mathbf{L}(\mathbf{x}) \rangle^{1/2}$ is non-zero and acts as a source in the vector perturbation equation.

To proceed one needs to obtain an expression for the Lorentz force in terms of the magnetic field characteristics. We assume that the magnetic field is Gaussian and satisfy [3],

$$\langle B_i(\mathbf{k}) B_j^\star(\mathbf{k}') \rangle = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') P_{ij}(\hat{\mathbf{k}}) M(k) \qquad \text{for} \qquad k \le k_D \;, \tag{6}$$

and vanishes for $k > k_D$. Here a star denotes complex conjugation, and $\delta(\mathbf{k} - \mathbf{k}')$ is the usual 3-dimensional Dirac delta function, and k_D is the magnetic field damping scale defined through the Alfvén waves damping [10, 26]. We approximate power spectrum $M(k)$ by simple power laws.

Following Ref. [11] it can be shown that the Fourier transform of the Lorentz force $L_i^{(V)}(\mathbf{k}) \equiv k\Pi_i(\mathbf{k})$ is related to the Fourier transform of spacial part of magnetic field energy momentum tensor $\tau_{mj}^{(B)}(\mathbf{k})$

$$\tau_{ij}^{(B)}(\mathbf{k}) = \frac{1}{4\pi}\frac{1}{(2\pi)^3}\int d^3\mathbf{q}[B_i(\mathbf{q})B_j(\mathbf{k}-\mathbf{q}) - \frac{1}{2}\delta_{ij}B_m(\mathbf{q})B_m(\mathbf{k}-\mathbf{q})] \;, \tag{7}$$

as

$$\Pi_i(\mathbf{k}) = \frac{1}{2}(P_{ij}(\hat{\mathbf{k}})\hat{k}_m + P_{im}(\hat{\mathbf{k}})\hat{k}_j)\tau_{mj}^{(B)}(\mathbf{k}) \;, \tag{8}$$

where $P_{ij}(\hat{\mathbf{k}}) = \delta_{ij} - \hat{k}_i\hat{k}_j$ is the transverse plane projector with unit wavenumber components $\hat{k}_i = k_i/k$.

Next we shall solve the Eq. (4). It can be solved in two different regimes, for length scales larger and smaller then comoving Silk scale λ_S. These two solutions are [11]:

⋆ For the scales $\lambda > \lambda_S$, $(k < k_S)$

$$\mathbf{\Omega}(\mathbf{k}, \eta) = \frac{k\mathbf{\Pi}(\mathbf{k})\eta}{(1+R)(\rho_{\gamma_0} + p_{\gamma_0})} \;, \tag{9}$$

where ρ_{γ_0} and p_{γ_0} are photon energy and pressure today.

⋆ For smaller scales with $\lambda < \lambda_S$, $(k > k_S)$

$$\mathbf{\Omega}(\mathbf{k}, \eta) = \frac{\mathbf{\Pi}(\mathbf{k})}{(kL_\gamma/5)(\rho_{\gamma_0} + p_{\gamma_0})} \;, \tag{10}$$

where L_γ is the comoving photon mean-free path.

Eqs. (9) and (10) define the vorticity through the magnetic source, $\mathbf{\Pi}(\mathbf{k})$, Eq. (8). As we can see the vorticity perturbations at scales below the Silk damping stay constant, while the vorticity perturbations above the Silk damping are increasing linearly with the time. In order to compute the magnetized vector mode induced effects we need to compute the vorticity perturbations two- and three-point correlations.

Below we present our computations. It must be stressed that the form of the magnetic field two-point correlation function, Eq. (6) presumes the following properties of the field: (i) transverse, divergence free field, $\nabla \cdot \mathbf{B} = 0$, and in the fourier space this is insured by the projector $P_{ij}(\hat{\mathbf{k}})$; (ii) isotropy (no preferred direction) insured by $\delta(\mathbf{k} - \mathbf{k}')$ and axis-symmetry of the field, the two point correlation is symmetric under under i and j indices exchanges; (iii) The magnetic field is a gaussianly distributed field.

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[3] For a vector field $\mathbf{F}$ we use

$$F_j(\mathbf{k}) = \int d^3\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}} F_j(\mathbf{x}), \qquad\qquad F_j(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} F_j(\mathbf{k}), \tag{5}$$

when Fourier transforming between real and wavenumber spaces; we assume flat spatial hypersurfaces.

A. Lorentz Force Two-Point Correlation Function

For the two-point correlation function of the $\mathbf{\Pi}(\mathbf{k})$ magnetic vector source we have

$$\zeta^{(12)}_{i_1 i_2}(\mathbf{k},\mathbf{k}') \equiv \langle \Pi^{\star}_{i_1}(\mathbf{k})\Pi_{i_2}(\mathbf{k}')\rangle \ . \tag{11}$$

Using results of Sect. A 2 one finds for it

$$\zeta^{(12)}_{i_1 i_2}(\mathbf{k},\mathbf{k}') = \delta(\mathbf{k}-\mathbf{k}')\psi^{(12)}_{i_1 i_2}(\mathbf{k}) \ , \tag{12}$$

with

$$\begin{aligned}\psi^{(12)}_{i_1 i_2}(\mathbf{k}) \ &= \ \frac{1}{(8\pi)^2}\int d^3\mathbf{q} M(|\mathbf{q}|)M(|\mathbf{k}-\mathbf{q}|) \\ & \left[P_{i_1 a}(\hat{\mathbf{k}})\hat{k}_b + P_{i_1 b}(\hat{\mathbf{k}})\hat{k}_a\right]\left[P_{i_2 c}(\hat{\mathbf{k}})\hat{k}_d + P_{i_2 d}(\hat{\mathbf{k}})\hat{k}_c\right]\left[P_{ac}(\hat{\mathbf{q}})P_{bd}(\widehat{\mathbf{k}-\mathbf{q}}) + P_{ad}(\hat{\mathbf{q}})P_{bc}(\widehat{\mathbf{k}-\mathbf{q}})\right] .\end{aligned} \tag{13}$$

Note that as usual we assume summation over the repeated indices. Using $P_{ij}(\hat{\mathbf{k}})$ projector symmetry properties and defining $\gamma = \hat{\mathbf{k}}\cdot\hat{\mathbf{q}}$, $\beta = \hat{\mathbf{k}}\cdot(\widehat{\mathbf{k}-\mathbf{q}})$, and $\mu = \hat{\mathbf{q}}\cdot(\widehat{\mathbf{k}-\mathbf{q}})$, after long but simple computations we obtain for the Lorentz force two-point correlation function

$$\begin{aligned}\langle L^{(V)\star}_i(\mathbf{k})L^{(V)}_j(\mathbf{k}')\rangle \ &= \ \frac{k^2 P_{ij}(\hat{\mathbf{k}})}{(8\pi)^2}\delta(\mathbf{k}-\mathbf{k}')\int d^3\mathbf{q} M(|\mathbf{q}|)M(|\mathbf{k}-\mathbf{q}|)(2-\beta^2-\gamma^2) + \\ &+ \ \frac{k_i k_j}{(8\pi)^2}\delta(\mathbf{k}-\mathbf{k}')\int d^3\mathbf{q} M(|\mathbf{q}|)M(|\mathbf{k}-\mathbf{q}|)\left[2\gamma^2\beta^2 - \gamma^2(1-\beta^2) - \beta^2(1-\gamma^2)\right] \\ &- \ \frac{k^2}{(8\pi)^2}\delta(\mathbf{k}-\mathbf{k}')\int d^3\mathbf{q} M(|\mathbf{q}|)M(|\mathbf{k}-\mathbf{q}|)\Big\{\hat{q}_i\hat{q}_j(1-\beta^2) + (\widehat{\mathbf{k}-\mathbf{q}})_i(\widehat{\mathbf{k}-\mathbf{q}})_j(1-\gamma^2) \\ &- \ \gamma(\hat{k}_i\hat{q}_j + \hat{k}_j\hat{q}_i)(1-2\beta^2) - \beta\left[\hat{k}_i(\widehat{\mathbf{k}-\mathbf{q}})_j + \hat{k}_j(\widehat{\mathbf{k}-\mathbf{q}})_i\right](1-2\gamma^2) \\ &- \ \gamma\beta\left[\hat{q}_i(\widehat{\mathbf{k}-\mathbf{q}})_j + \hat{q}_j(\widehat{\mathbf{k}-\mathbf{q}})_i\right]\Big\} .\end{aligned} \tag{14}$$

Note that the form of the Lorentz force two-point correlation function, Eq. (14), has several symmetries: (i) symmetry under i and j index exchange; (ii) the symmetry under $\hat{\mathbf{q}}$ and $\widehat{\mathbf{k}-\mathbf{q}}$ exchange (i.e. γ and β angles exchange symmetry). It must be underlined that the trace of Eq. (14) is given by

$$\langle L^{(V)\star}_i(\mathbf{k})L^{(V)}_i(\mathbf{k}')\rangle = \frac{k^2}{2(4\pi)^2}\delta(\mathbf{k}-\mathbf{k}')\int d^3\mathbf{q} M(|\mathbf{q}|)M(|\mathbf{k}-\mathbf{q}|)(1+\mu\gamma\beta-2\beta^2\gamma^2) \ , \tag{15}$$

which totally agrees with the result of Refs. [8, 11, 12]. Further simplification of Eq. (14) gives following result for the Lorentz force two-point correlation function:

$$\langle L^{(V)\star}_i(\mathbf{k})L^{(V)}_j(\mathbf{k}')\rangle = \frac{k^2}{(8\pi)^2}P_{ij}(\hat{\mathbf{k}})\delta(\mathbf{k}-\mathbf{k}')\int d^3\mathbf{q} M(|\mathbf{q}|)M(|\mathbf{k}-\mathbf{q}|)(1+\mu\gamma\beta-2\beta^2\gamma^2) \ . \tag{16}$$

The ensemble averaging procedure insures that rotational isotropy is preserved. Note that without ensemble averaging *in one realization* the isotropy of the two point correlation can be violated and only the averaging leads to cancelation of anisotropic component and restoration of isotropy.

B. Lorentz Force Three-Point Correlation Function

The three-point correlation function of the vector magnetic source is given[4] by

$$\zeta^{(123)}_{i_1 i_2 i_3}(\mathbf{k_1},\mathbf{k_2},\mathbf{k_3}) = \langle \Pi_{i_1}(\mathbf{k_1})\Pi_{i_2}(\mathbf{k_2})\Pi_{i_3}(\mathbf{k_3})\rangle \ . \tag{17}$$

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[4] It is appropriate to define the three point correlation function as an average of three Π_i's without the complex conjugation.

Using results of Sec. A 2 one finds

$$\zeta^{(123)}_{i_1 i_2 i_3}(\mathbf{k_1},\mathbf{k_2},\mathbf{k_3}) = \delta(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})\psi^{(123)}_{i_1 i_2 i_3}(\mathbf{k_1},\mathbf{k_2}) \ , \tag{18}$$

with

$$\begin{aligned}\psi^{(123)}_{i_1 i_2 i_3}(\mathbf{k_1},\mathbf{k_2}) \ = \ & \frac{1}{8\pi^3}\int d^3\mathbf{q} M(|\mathbf{q}|)M(|\mathbf{k}_1-\mathbf{q}|)M(|\mathbf{k}_2+\mathbf{q}|) \\ & P_{i_1 j_1}(\hat{\mathbf{k}}_1)P_{j_1 s_3}(\mathbf{k_1}\hat{-}\mathbf{q})(\mathbf{k_1}\hat{+}\mathbf{k_2})_{s_3}P_{i_2 j_2}(\hat{\mathbf{k}}_2)P_{j_2 j_3}(\mathbf{k}_2\hat{+}\mathbf{q})P_{i_3 j_3}(\mathbf{k_1}\hat{+}\mathbf{k_2})\hat{k}_{1s_1}P_{s_1 s_2}(\hat{\mathbf{q}})\hat{k}_{2s_2} \ .\end{aligned} \tag{19}$$

The Lorentz force three point correlation function is given as

$$\begin{aligned}\langle L^{(V)}_{i_1}(\mathbf{k_1})L^{(V)}_{i_2}(\mathbf{k_2})L^{(V)}_{i_3}(\mathbf{k_3})\rangle \ = \ & \frac{1}{8\pi^3}\delta(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})\int d^3\mathbf{q} M(|\mathbf{q}|)M(|\mathbf{k}_1-\mathbf{q}|)M(|\mathbf{k}_2+\mathbf{q}|) \\ & P_{i_1 j_1}(\hat{\mathbf{k}}_1)P_{i_2 j_2}(\hat{\mathbf{k}}_2)P_{i_3 j_3}(\mathbf{k_1}\hat{+}\mathbf{k_2})P_{j_1 s_3}(\mathbf{k_1}\hat{-}\mathbf{q})(\mathbf{k_1}+\mathbf{k_2})_{s_3}P_{j_2 j_3}(\mathbf{k}_2\hat{+}\mathbf{q}) \\ & [(\mathbf{k_1}\times\hat{\mathbf{q}})(\mathbf{k_2}\times\hat{\mathbf{q}})] \ .\end{aligned} \tag{20}$$

We see that the r.h.s. of Eq. (20) is symmetric under exchange $\mathbf{k_1}$ and $\mathbf{k_2}$. It is obvious that this symmetry is reflected in the CMB temperature three point correlation function, see below.

III. TEMPERATURE FLUCTUATIONS FROM THE MAGNETIZED VECTOR MODE

Vector perturbations induce CMB temperature anisotropies via the Doppler and integrated Sachs-Wolfe effects [27, 28],

$$\frac{\Delta T}{T}(\mathbf{x_0},\mathbf{n},\eta_0) = -\mathbf{v}\cdot\mathbf{n}|^{\eta_0}_{\eta_{\rm dec}} + \int_{\eta_{\rm dec}}^{\eta_0} d\eta\, \dot{\mathbf{V}}\cdot\mathbf{n}, \tag{21}$$

where $\mathbf{n}$ is the unit vector in the light propagation direction, $\eta_{\rm dec}$ is the conformal time at decoupling. $\mathbf{x_0}$ denotes the observer position, $\mathbf{x}_{\rm dec} = \mathbf{x_0} + \mathbf{n}(\eta_0 - \eta_{\rm dec})$. Due to the spherical symmetry the temperature fluctuations are decomposed using the spherical harmonics as,

$$\frac{\Delta T}{T}(\mathbf{x_0},\mathbf{n},\eta_\mathbf{0}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}(\mathbf{x_0},\eta_0)\cdot Y_{lm}(\mathbf{n}) \ . \tag{22}$$

Accounting for the spherical harmonics property, see Chapter 5 of Ref. [29], we obtain $a_{lm}(\mathbf{x_0},\eta_0) = \int d\Omega_\mathbf{n} Y^\star_{lm}(\mathbf{n})\Delta T/T(\mathbf{x_0},\mathbf{n},\eta_0)$. The decaying nature of the vector potential $\mathbf{V}$ implies that most of its contribution toward the integrated Sachs-Wolfe term comes from near $\eta_{\rm dec}$. Neglecting a possible dipole contribution due to $\mathbf{v}$ today, from Eq. (21) we obtain,

$$\frac{\Delta T}{T}(\mathbf{n}) \simeq \mathbf{v}(\eta_{\rm dec})\cdot\mathbf{n} - \mathbf{V}(\eta_{\rm dec})\cdot\mathbf{n} = \mathbf{\Omega}(\eta_{\rm dec})\cdot\mathbf{n} \ . \tag{23}$$

Here we placed observer at the origin $\mathbf{x_0} = 0$ and we skip η_0 denoting $\Delta T/T(\mathbf{n}) \equiv \Delta T/T(\mathbf{x_0} = 0, \mathbf{n}, \eta = \eta_0)$ and $a_{lm} \equiv a_{lm}(\mathbf{x_0} = 0, \eta_0)$. Since $\mathbf{x}_{\rm dec} = \mathbf{n}(\eta_0 - \eta_{\rm dec})$ Fourier transform of Eq. (23) results in

$$\frac{\Delta T}{T}(\mathbf{k},\mathbf{n}) = \mathbf{\Omega}(\mathbf{k},\eta_{\rm dec})\cdot\mathbf{n})e^{i(\mathbf{k}\cdot\mathbf{n})\Delta\eta} \ , \tag{24}$$

where $\mathbf{\Omega}(\mathbf{k},\eta_{\rm dec})$ is the Fourier amplitude of vorticity perturbations at $\eta = \eta_{\rm dec}$, wave vector $\mathbf{k} = k\hat{\mathbf{k}}$ labels the resulting Fourier mode after transforming from the coordinate representation $\mathbf{x}$ to the momentum representation by using $e^{i\mathbf{kx}}$, and $\Delta\eta = \eta_0 - \eta_{\rm dec} \approx \eta_0$ is the conformal time from decoupling until today.

For the multipole coefficient Fourier transform we obtain,

$$a_{lm}(\mathbf{k}) = \int d\mathbf{n}(\mathbf{\Omega}(\mathbf{k},\eta_{\rm dec})\cdot\mathbf{n})e^{i\mathbf{k}\cdot\mathbf{n}\eta_0}Y^\star_{lm}(\hat{\mathbf{n}}) \ , \tag{25}$$

where we use $a_{lm}(\mathbf{k}) \equiv a_{lm}(\mathbf{k},\eta = \eta_0)$.

A. Two-point correlation function

The two-point correlation of the temperature fluctuations is given by

$$\mathcal{C}(\mathbf{n_1},\mathbf{n_2}) \equiv \langle \frac{\Delta T}{T}(\mathbf{n_1})\frac{\Delta T}{T}(\mathbf{n_2})\rangle = \sum_{l_1 m_1}\sum_{l_2 m_2}\langle a^\star_{l_1 m_1} a_{l_2 m_2}\rangle Y^\star_{l_1 m_1}(\mathbf{n_1}) Y_{l_2 m_2}(\mathbf{n_2}) \ , \tag{26}$$

here $\sum_{lm} \equiv \sum_{l=0}^{\infty}\sum_{m=-l}^{l}$. In the case when the spatial isotropy and rotational invariance is preserved the multipoles with different l and m do not correlate and $\langle a^\star_{l_1 m_1} a_{l_2 m_2}\rangle = C_{l_1}\delta_{l_1 l_2}\delta_{m_1 m_2}$. It is obvious that $\mathcal{C}(\mathbf{n_1},\mathbf{n_2})$ is the function of the angle between $\mathbf{n_1}$ and $\mathbf{n_2}$ vectors, and we can easy get

$$\mathcal{C}(\mathbf{n_1},\mathbf{n_2}) = \sum_{lm}\frac{2l+1}{4\pi} C_l P_l(\mathbf{n_1}\cdot\mathbf{n_2}). \tag{27}$$

The sound waves (density perturbations) sourced by the magnetic field keep the rotational invariance unchanged and thus the multipole correlation matrix has a diagonal form, and the two-point correlation function depends on one angle, $\mathbf{n_1}\cdot\mathbf{n_2}$. The same time the temperature fluctuations are non-gaussian [6, 7] due to the non-linearity of the magnetic field energy density with respect to the magnetic field. Note, that the magnetic field which has a non-gaussian energy momentum tensor [8] can be itself field with Gaussian distribution.

As soon as the vector mode is present in the Universe there is an additional direction inserted through the vorticity field $\mathbf{\Omega}(\eta_{\rm dec})$. Before ensemble averaging procedure the two-point correlation is not rotationally invariant and the cross correlation function between the multipoles has off-diagonal components [27, 28]. The measurement of these off-diagonal terms might serve as a tool to constraint the primordial homogeneous magnetic field.

We define the multipole two point correlation function as usual:

$$D^{m_1 m_2}_{l_1 l_2} \equiv \langle a^\star_{l_1 m_1} a_{l_2 m_2}\rangle = \int \frac{d^3\mathbf{k}_1}{(2\pi)^3}\frac{d^3\mathbf{k}_2}{(2\pi)^3}\langle a^\star_{l_1 m_1}(\mathbf{k}_1) a_{l_2 m_2}(\mathbf{k}_2)\rangle \ , \tag{28}$$

and the simple calculation leads to

$$\langle a^\star_{l_1 m_1} a_{l_2 m_2}\rangle = \int \frac{d^3\mathbf{k_1}}{(2\pi)^3}\frac{d^3\mathbf{k}_2}{(2\pi)^3} d\mathbf{n}_1 d\mathbf{n}_2 e^{i(-\mathbf{k_1}\cdot\mathbf{n_1}+\mathbf{k_2}\cdot\mathbf{n_2})\eta_0} Y^\star_{l_1 m_1}(\mathbf{n}_1) Y_{l_2 m_2}(\mathbf{n}_2)\langle[\mathbf{n_1}\cdot\mathbf{\Omega}^\star(\mathbf{k_1},\eta_{\rm dec})][\mathbf{n_2}\cdot\mathbf{\Omega}(\mathbf{k_2},\eta_{\rm dec})]\rangle . \tag{29}$$

We also decompose the vorticity perturbation plane wave over the vector spherical harmonics as, $\mathbf{\Omega}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{n}\eta_0} = \sum_{\lambda,l,m} A^{(\lambda)}_{lm}\mathbf{Y}^{(\lambda)}_{lm}(\mathbf{n})$, with $\lambda = -1, 0, 1$ and use $\nabla\cdot\mathbf{\Omega} = 0$ condition, leading to $\mathbf{\Omega}(\mathbf{k})\cdot\mathbf{Y}^{(-1)}_{lm}(\hat{\mathbf{k}}) = 0$, see also Appendix of Ref. [28]. We use also the definition $\mathbf{n}\cdot Y^\star_{lm}(\mathbf{n}) = \mathbf{Y}^{(-1)\star}_{lm}(\mathbf{n})$. Then we have

$$D^{m_1 m_2}_{l_1 l_2} = \int \frac{d^3\mathbf{k}_1}{(2\pi)^3}\frac{d^3\mathbf{k}_2}{(2\pi)^3} d\mathbf{n}_1 d\mathbf{n}_2 Y^\star_{l_1 m_1}(\mathbf{n}_1) Y_{l_2 m_2}(\mathbf{n}_2)\sum_{t_1 r_1}\sum_{t_2 r_2} Y^\star_{t_1 r_1}(\mathbf{n}_1) Y_{t_2 r_2}(\mathbf{n}_2)\langle A^{(-1)\star}_{t_1 r_1} A^{(-1)}_{t_2 r_2}\rangle \ , \tag{30}$$

with

$$A^{(-1)}_{lm} = 4\pi i^{l-1}\sqrt{l(l+1)}\frac{j_l(k\eta_0)}{k\eta_0}(\mathbf{\Omega}(\mathbf{k})\cdot\mathbf{Y}^{(+1)}_{lm}(\hat{\mathbf{k}})) \ . \tag{31}$$

The integration over $\mathbf{n}$ in Eq. (30) gives us $\delta_{l_1 t_1}$, $\delta_{l_2 t_2}$, $\delta_{m_1 r_1}$, $\delta_{m_2 r_2}$, and the sums are eliminated and we get

$$\begin{aligned} D^{m_1 m_2}_{l_1 l_2} &= \frac{(-1)^{l_1 - l_2}\ i^{l_1+l_2-2}\sqrt{l_1(l_1+1)l_2(l_2+1)}}{(2\pi^2)^2}\int d^3\mathbf{k}_1\ d^3\mathbf{k}_2 \frac{j_{l_1}(k_1\eta_0) j_{l_2}(k_2\eta_0)}{k_1 k_2 \eta_0^2} \\ &\quad \left[\mathbf{Y}^{(+1)\star}_{l_1 m_1}(\hat{\mathbf{k}}_1)\right]_{i_1}\left[\mathbf{Y}^{(+1)}_{l_2 m_2}(\hat{\mathbf{k}}_2)\right]_{i_2}\langle\Omega^\star_{i_1}(\mathbf{k}_1)\Omega_{i_2}(\mathbf{k_2})\rangle \ . \end{aligned} \tag{32}$$

Next we have to consider separately case of large scale approximation Eq. (9) and case of small scale approximation Eq. (10). Using Eq. (16) we obtain:

$$\begin{aligned} D^{m_1 m_2}_{l_1 l_2} &= \frac{(-1)^{l_1 - l_2}\ i^{l_1+l_2-2}\sqrt{l_1(l_1+1)l_2(l_2+1)}}{(2\pi)^3[2(1+R_{\rm dec})(\rho_{\gamma_0}+p_{\gamma_0})]^2}\left(\frac{\eta_{\rm dec}}{\eta_0}\right)^2\int d^3\mathbf{k}\ d^3\mathbf{q} M(|\mathbf{q}|)M(|\mathbf{k}-\mathbf{q}|) j_{l_1}(k\eta_0) j_{l_2}(k\eta_0) \\ &\quad (1+\mu\gamma\beta - 2\gamma^2\beta^2)\left(\mathbf{Y}^{(+1)\star}_{l_1 m_1}(\hat{\mathbf{k}})\cdot\mathbf{Y}^{(+1)}_{l_2 m_2}(\hat{\mathbf{k}})\right) \end{aligned} \tag{33}$$

for the scales larger then the Silk damping scale. For the scale smaller than the Silk damping scale we have

$$D_{l_1 l_2}^{m_1 m_2} = \frac{(-1)^{l_1-l_2}\ i^{l_1+l_2-2}\sqrt{l_1(l_1+1)l_2(l_2+1)}}{(2\pi)^3[2(L_\gamma/5)(\rho_{\gamma_0}+p_{\gamma_0})]^2}\int d^3\mathbf{k}\ d^3\mathbf{q} M(|\mathbf{q}|)M(|\mathbf{k}-\mathbf{q}|)\frac{j_{l_1}(k\eta_0)j_{l_2}(k\eta_0)}{(k\eta_0)^2}$$
$$(1+\mu\gamma\beta-2\gamma^2\beta^2)\left(\mathbf{Y}_{l_1 m_1}^{(+1)\star}(\hat{\mathbf{k}})\cdot\mathbf{Y}_{l_2 m_2}^{(+1)}(\hat{\mathbf{k}})\right) . \tag{34}$$

Recall that we assume that the magnetic field power spectrum is given by a simple power law, $M(|\mathbf{q}|)\propto q^n$. To proceed we have to evaluate the following integrals over angular variables

$$\mathcal{I}_{l_1,l_2}^{m_1,m_2} = \int d\Omega_{\hat{\mathbf{k}}}\int d\Omega_{\hat{\mathbf{q}}}(1+\mu\beta\gamma-2\gamma^2\beta^2)(k^2+q^2-2kq\gamma)^{n/2}\left(\mathbf{Y}_{l_1 m_1}^{(+1)\star}(\hat{\mathbf{k}})\cdot\mathbf{Y}_{l_2 m_2}^{(+1)}(\hat{\mathbf{k}})\right) . \tag{35}$$

It is easy to see that integrals over $d\Omega_{\hat{\mathbf{k}}}$ and $d\Omega_{\hat{\mathbf{q}}}$ separate. First we evaluate the integral over $d\Omega_{\hat{\mathbf{q}}}$. For a particular $\hat{\mathbf{k}}$, choose the polar axis for the $d\Omega_{\hat{\mathbf{q}}}$ integral in the $\mathbf{z}$ direction. Since $\gamma=\cos\theta_{\hat{\mathbf{q}}}$ the integrand does not depend on the azimuthal angle $\phi_{\hat{\mathbf{q}}}$ and the integration over $\phi_{\hat{\mathbf{q}}}$ simply gives 2π. The integration over $d\cos\theta_{\hat{\mathbf{q}}}$ is simple to be evaluated and is given as $\int_{-1}^{1} d\gamma(1-\gamma^2)\left(1-\frac{2q\gamma(k+q\gamma)}{k^2+q^2-2kq\gamma}\right)\left(k^2+q^2-2kq\gamma\right)^{n/2}$, with $q=|\mathbf{q}|$. Thus

$$\mathcal{I}_{l_1,l_2}^{m_1,m_2} = 2\pi\delta_{l_1 l_2}\delta_{m_1 m_2}\int_{-1}^{1} d\gamma(1-\gamma^2)\left(1-\frac{2q\gamma(k+q\gamma)}{k^2+q^2-2kq\gamma}\right)\left(k^2+q^2-2kq\gamma\right)^{n/2} \tag{36}$$

and finally only the diagonal cross correlations are present. Note that, the rotational symmetry (absence of off-diagonal terms in $D_{l_1 l_2}^{m_1 m_2}$) is due to the ensemble averaging.

B. Bispectrum definition and calculation

The standard approach to study the CMB non-gaussianity consists on the CMB temperature fluctuations investigation. Below we present the self-consistent way to describe the bispectrum. As usual the three-point correlation function of CMB temperature anisotropy is defined as

$$\xi(\mathbf{n_1},\mathbf{n_2},\mathbf{n_3}) \equiv \langle\frac{\Delta T}{T}(\mathbf{n_1})\frac{\Delta T}{T}(\mathbf{n_2})\frac{\Delta T}{T}(\mathbf{n_3})\rangle = \sum_{l_i m_i}\langle a_{l_1 m_1}a_{l_2 m_2}a_{l_3 m_3}\rangle Y_{l_1 m_1}(\mathbf{n_1})Y_{l_2 m_2}(\mathbf{n_2})Y_{l_3 m_3}(\mathbf{n_3})\ , i=1,2,3\ , \tag{37}$$

In order to proceed we need to calculate the following form

$$B_{l_1 l_2 l_3}^{m_1 m_2 m_3} \equiv \langle a_{l_1 m_1}a_{l_2 m_2}a_{l_3 m_3}\rangle = \int\frac{d^3\mathbf{k}_1}{(2\pi)^3}\frac{d^3\mathbf{k}_2}{(2\pi)^3}\frac{d^3\mathbf{k}_3}{(2\pi)^3}\langle a_{l_1 m_1}(\mathbf{k}_1)a_{l_2 m_2}(\mathbf{k}_2)a_{l_3 m_3}(\mathbf{k}_3)\rangle . \tag{38}$$

Using expression for a_{lm} given by Eq. (25) we get

$$\langle a_{l_1 m_1}a_{l_2 m_2}a_{l_3 m_3}\rangle = \int\frac{d^3\mathbf{k}_1}{(2\pi)^3}\frac{d^3\mathbf{k}_2}{(2\pi)^3}\frac{d^3\mathbf{k}_3}{(2\pi)^3}d\mathbf{n}_1 d\mathbf{n}_2 d\mathbf{n}_3 e^{i(\mathbf{k_1}\cdot\mathbf{n_1}+\mathbf{k_2}\cdot\mathbf{n_2}+\mathbf{k_3}\cdot\mathbf{n_3})\Delta\eta}$$
$$Y_{l_1 m_1}^\star(\mathbf{n}_1)Y_{l_2 m_2}^\star(\mathbf{n}_2)Y_{l_3 m_3}^\star(\mathbf{n}_3)n_{1i_1}n_{2i_2}n_{3i_3}\langle\Omega_{i_1}(\mathbf{k_1},\eta_{\rm dec})\Omega_{i_2}(\mathbf{k_2},\eta_{\rm dec})\Omega_{i_3}(\mathbf{k_3},\eta_{\rm dec})\rangle . \tag{39}$$

First we integrate over $d\mathbf{n}_i$ by using Eq. (117) on p.227 [29]. Proceeding in the way given in Sec. IIIA we arrive at

$$B_{l_1 l_2 l_3}^{m_1 m_2 m_3} = \frac{i^{l_1+l_2+l_3-3}}{(2\pi^2)^3}\sqrt{l_1(l_1+1)l_2(l_2+1)l_3(l_3+1)}\int d^3\mathbf{k_1}d^3\mathbf{k_1}d^3\mathbf{k_3}\frac{j_{l_1}(k_1\eta_0)j_{l_2}(k_2\eta_0)j_{l_3}(k_3\eta_0)}{k_1 k_2 k_3 {\eta_0}^3}$$
$$\mathbf{Y}_{l_1 m_1}^{(+1)\,\star}(\hat{\mathbf{k}}_1)|_{i_1}\mathbf{Y}_{l_2 m_2}^{(+1)\,\star}(\hat{\mathbf{k}}_2)|_{i_2}\mathbf{Y}_{l_3 m_3}^{(+1)\,\star}(\hat{\mathbf{k}}_3)|_{i_3}\langle\Omega_{i_1}(\mathbf{k_1},\eta_{\rm dec})\Omega_{i_2}(\mathbf{k_2},\eta_{\rm dec})\Omega_{i_3}(\mathbf{k_3},\eta_{\rm dec})\rangle . \tag{40}$$

Next we have to consider separately case of large scale approximation Eq. (9) and case of small scale approximation Eq. (10). Using Eqs. (17, 18, 19), relation Eq. (A8) and the representation for the Dirac delta function Eq. (A9) after some computations for the large scale approximation ($L>L_S$) we obtain

$$B_{l_1 l_2 l_3}^{m_1 m_2 m_3} = \frac{i^{l_1+l_2+l_3-3}\sqrt{l_1(l_1+1)l_2(l_2+1)l_3(l_3+1)}}{(2\pi^3)^3[(1+R_{\rm dec})(\rho_{\gamma_0}+p_{\gamma_0})]^3}\left(\frac{\eta_{\rm dec}}{\eta_0}\right)^3\int k_1^2 dk_1 k_2^2 dk_2 k_3^2 dk_3 j_{l_1}(k_1\eta_0)j_{l_2}(k_2\eta_0)j_{l_3}(k_3\eta_0)$$

$$\int d\Omega_{\hat{\mathbf{k}}_1} d\Omega_{\hat{\mathbf{k}}_2} d\Omega_{\hat{\mathbf{k}}_3} \int d\Omega_{\hat{\mathbf{q}}} \int q^2 dq M(|\mathbf{q}|) M(|\mathbf{k}_1 - \mathbf{q}|) M(|\mathbf{k}_2 + \mathbf{q}|)$$
$$\sum_{t_1 r_1} \sum_{t_2 r_2} \sum_{t_3 r_3} i^{t_1+t_2+t_3} \int x^2 dx j_{t_1}(k_1 x) j_{t_2}(k_2 x) j_{t_3}(k_3 x)\ G^{t_1 t_2 t_3}_{r_1 r_2 r_3}$$
$$\left[\mathbf{Y}^{(-1)}_{t_1 r_1}(\hat{\mathbf{k}}_1) \times \hat{\mathbf{q}}\right]_a \left[\mathbf{Y}^{(1)}_{l_1 m_1}{}^{\star}(\hat{\mathbf{k}}_1) \times (\mathbf{k}_1 \hat{-} \mathbf{q})\right]_b$$
$$\left[\mathbf{Y}^{(-1)}_{t_2 r_2}(\hat{\mathbf{k}}_2) \times \hat{\mathbf{q}}\right]_a \left[\mathbf{Y}^{(1)}_{l_2 m_2}{}^{\star}(\hat{\mathbf{k}}_2) \times (\mathbf{k}_2 \hat{+} \mathbf{q})\right]_c$$
$$\left[\mathbf{Y}^{(-1)}_{t_3 r_3}(\hat{\mathbf{k}}_3) \times (\mathbf{k}_1 \hat{-} \mathbf{q})\right]_b \left[\mathbf{Y}^{(1)}_{l_3 m_3}{}^{\star}(\hat{\mathbf{k}}_3) \times (\mathbf{k}_2 \hat{+} \mathbf{q})\right]_c , \tag{41}$$

where $G^{t_1 t_2 t_3}_{r_1 r_2 r_3}$ is the usual Gaunt integral,

$$G^{t_1 t_2 t_3}_{r_1 r_2 r_3} = \int d\Omega_{\hat{\mathbf{n}}} Y_{t_1 r_1}(\hat{\mathbf{n}}) Y_{t_2 r_2}(\hat{\mathbf{n}}) Y_{t_3 r_3}(\hat{\mathbf{n}}) . \tag{42}$$

The indices $a,\ b,\ c$ correspond to the vector components and repeated ones reflect the scalar product of the corresponding vectors.

Similarly for the small scale approximation ($L < L_S$) we obtain

$$B^{m_1 m_2 m_3}_{l_1 l_2 l_3} = \frac{i^{l_1+l_2+l_3-3}\sqrt{l_1(l_1+1) l_2(l_2+1) l_3(l_3+1)}}{(2\pi^3)^3[(L_\gamma/5)(\rho_{\gamma 0} + p_{\gamma 0})]^3 {\eta_0}^3} \int dk_1 dk_2 dk_3 j_{l_1}(k_1\eta_0) j_{l_2}(k_2\eta_0) j_{l_3}(k_3\eta_0)$$
$$\int d\Omega_{\hat{\mathbf{k}}_1} d\Omega_{\hat{\mathbf{k}}_2} d\Omega_{\hat{\mathbf{k}}_3} \int d\Omega_{\hat{\mathbf{q}}} \int q^2 dq M(|\mathbf{q}|) M(|\mathbf{k}_1 - \mathbf{q}|) M(|\mathbf{k}_2 + \mathbf{q}|)$$
$$\sum_{t_1 r_1} \sum_{t_2 r_2} \sum_{t_3 r_3} i^{t_1+t_2+t_3} \int x^2 dx j_{t_1}(k_1 x) j_{t_2}(k_2 x) j_{t_3}(k_3 x)\ G^{t_1 t_2 t_3}_{r_1 r_2 r_3}$$
$$\left[\mathbf{Y}^{(-1)}_{t_1 r_1}(\hat{\mathbf{k}}_1) \times \hat{\mathbf{q}}\right]_a \left[\mathbf{Y}^{(1)}_{l_1 m_1}{}^{\star}(\hat{\mathbf{k}}_1) \times (\mathbf{k}_1 \hat{-} \mathbf{q})\right]_b$$
$$\left[\mathbf{Y}^{(-1)}_{t_2 r_2}(\hat{\mathbf{k}}_2) \times \hat{\mathbf{q}}\right]_a \left[\mathbf{Y}^{(1)}_{l_2 m_2}{}^{\star}(\hat{\mathbf{k}}_2) \times (\mathbf{k}_2 \hat{+} \mathbf{q})\right]_c$$
$$\left[\mathbf{Y}^{(-1)}_{t_3 r_3}(\hat{\mathbf{k}}_3) \times (\mathbf{k}_1 \hat{-} \mathbf{q})\right]_b \left[\mathbf{Y}^{(1)}_{l_3 m_3}{}^{\star}(\hat{\mathbf{k}}_3) \times (\mathbf{k}_2 \hat{+} \mathbf{q})\right]_c . \tag{43}$$

Using the Wigner D functions and proceeding in the way analogous to Sec. IIIA it can be shown that there are non-zero cross correlations between non-equal l and m. Again the proper answer assumes accounting for the angular dependence in the power spectra M.

IV. CONCLUDING REMARKS

We present the study of the magnetized perturbation vector mode induced CMB two- and three-point correlation functions in a common framework. We show that already the vorticity two-point correlation functions reflect the anisotropy of the considered perturbations before ensemble averaging. One of the implications of our results is that CMB bispectrum computation technique presented in Ref. [30] in the case of magnetized perturbations should be applied with caution.

Note that in the present work we have focused on derivation of main analytic results for the CMB two- and three-point correlation functions arising from the vector mode supported by the stochastic cosmological magnetic field. We plan to present a detailed analysis of obtained equations and phenomenological estimates in a separate publication.

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Appendix A: Useful Mathematical Formulae

In this Appendix we list various mathematical results we use in the computations.

1. Wigner D Functions

Wigner D functions relate helicity basis vectors $\mathbf{e}'_{\pm 1} = \mp(\mathbf{e}_\Theta \pm i\mathbf{e}_\phi)/\sqrt{2}$ and $\mathbf{e}'_0 = \mathbf{e}_r$ to spherical basis vectors $\mathbf{e}_{\pm 1} = \mp(\mathbf{e}_x \pm i\mathbf{e}_y)/\sqrt{2}$ and $\mathbf{e}_0 = \mathbf{e}_z$ (see Eq. (53), p. 11, [29]) through

$$\mathbf{e}'_\mu = \sum_\nu D^1_{\nu\mu}(\phi, \Theta, 0)\mathbf{e}_\nu, \qquad \nu, \mu = -1, 0, 1 \; . \tag{A1}$$

In both the spherical basis and the helicity basis the following relations hold: $\mathbf{e}_\nu \mathbf{e}^\mu = \delta_{\nu\mu}$, $\mathbf{e}^\mu = (-1)^\mu \mathbf{e}_{-\mu}$, $\mathbf{e}^\mu = \mathbf{e}^\star_\mu$, $\mathbf{e}_\mu \times \mathbf{e}_\nu = -i\epsilon_{\mu\nu\lambda}\mathbf{e}_\lambda$.

2. Calculation of $\mathcal{B}$

For calculation of the two-point correlation function we must determine the magnetic field energy momentum two-point cross correlation $\langle \tau^{(B)}_{ab}(\mathbf{k})\tau^{(B)}_{cd}(\mathbf{k}')\rangle$ given by Eq. A6 of [11] and Eq. 4.1 of [8]. It is easy to show that the parts of the $\tau^{(B)}_{ab}$ (and $\tau^{(B)}_{cd}$ proportional to the δ_{cd}), see Eq. (7), do not contribute to the integral of Eq. (8). Then we need to compute the following object

$$\mathcal{B}_{abcd}(\mathbf{k_1}, \mathbf{k_2}) = \int \frac{d^3\mathbf{q}_1}{(2\pi)^3}\frac{d^3\mathbf{q}_2}{(2\pi)^3}\langle B^\star_a(\mathbf{q}_1)B^\star_b(\mathbf{k}_1 - \mathbf{q}_1)B_c(\mathbf{q}_2)B_d(\mathbf{k}_2 - \mathbf{q}_2)\rangle \; . \tag{A2}$$

Using Eq. (6) and Wick's theorem, we obtain that the contribution of the two-point correlation of the magnetic field energy momentum into the vorticity perturbation is given by

$$\frac{\delta(\mathbf{k} - \mathbf{k}')}{(4\pi)^2}\int d^3q M(|\mathbf{q}|)M(|\mathbf{k} - \mathbf{q}|)\left[P_{ac}(\hat{\mathbf{q}})P_{bd}(\widehat{\mathbf{k} - \mathbf{q}}) + P_{ad}(\hat{\mathbf{q}})P_{bc}(\widehat{\mathbf{k} - \mathbf{q}})\right] \; . \tag{A3}$$

For calculation of bispectrum we need to know following object

$$\mathcal{B}_{abcdef}(\mathbf{k_1}, \mathbf{k_2}, \mathbf{k_3}) = \int \frac{d^3\mathbf{q}_1}{(2\pi)^3}\frac{d^3\mathbf{q}_2}{(2\pi)^3}\frac{d^3\mathbf{q}_3}{(2\pi)^3}\langle B_a(\mathbf{q}_1)B_b(\mathbf{k}_1 - \mathbf{q}_1)B_c(\mathbf{q}_2)B_d(\mathbf{k}_2 - \mathbf{q}_2)B_e(\mathbf{q}_3)B_f(\mathbf{k}_3 - \mathbf{q}_3)\rangle \; . \tag{A4}$$

Assuming that the magnetic field obeys Gaussian statistics we can expand the six-point correlation function using Wick's theorem. Doing this we will get seven terms proportional to either $\delta(\mathbf{k_1})$, $\delta(\mathbf{k_2})$ or $\delta(\mathbf{k_3})$ (which we neglect) and eight terms proportional to $\delta(\mathbf{k_1} + \mathbf{k_2} + \mathbf{k_3})$ which we keep. So, for $\mathcal{B}_{abcdef}$ one finds [8, 31]:

$$\begin{aligned}
\mathcal{B}_{abcdef}(\mathbf{k_1}, \mathbf{k_2}, \mathbf{k_3}) \;=\;& \delta(\mathbf{k_1} + \mathbf{k_2} + \mathbf{k_3})\int d^3\mathbf{q} M(|\mathbf{q}|)M(|\mathbf{k}_1 - \mathbf{q}|)M(|\mathbf{k}_2 + \mathbf{q}|) \\
& \{P_{ac}(\mathbf{q})P_{be}(\mathbf{k}_1 - \mathbf{q})P_{df}(\mathbf{k}_2 + \mathbf{q}) + P_{ac}(\mathbf{q})P_{bf}(\mathbf{k}_1 - \mathbf{q})P_{de}(\mathbf{k}_2 + \mathbf{q}) \\
+\;& P_{ad}(\mathbf{q})P_{be}(\mathbf{k}_1 - \mathbf{q})P_{ef}(\mathbf{k}_2 + \mathbf{q}) + P_{ad}(\mathbf{q})P_{bf}(\mathbf{k}_1 - \mathbf{q})P_{ce}(\mathbf{k}_2 + \mathbf{q})\} \\
+\;& \delta(\mathbf{k_1} + \mathbf{k_2} + \mathbf{k_3})\int d^3\mathbf{q} M(|\mathbf{q}|)M(|\mathbf{k}_1 - \mathbf{q}|)M(|\mathbf{k}_3 + \mathbf{q}|) \\
& \{P_{ae}(\mathbf{q})P_{bc}(\mathbf{k}_1 - \mathbf{q})P_{df}(\mathbf{k}_3 + \mathbf{q}) + P_{ae}(\mathbf{q})P_{bd}(\mathbf{k}_1 - \mathbf{q})P_{df}(\mathbf{k}_3 + \mathbf{q}) \\
+\;& P_{af}(\mathbf{q})P_{bc}(\mathbf{k}_1 - \mathbf{q})P_{de}(\mathbf{k}_3 + \mathbf{q}) + P_{af}(\mathbf{q})P_{bd}(\mathbf{k}_1 - \mathbf{q})P_{ce}(\mathbf{k}_3 + \mathbf{q})\} \; .
\end{aligned} \tag{A5}$$

Assuming that projector $\mathcal{A}^{abcdef}$ acting on this object is symmetric w.r.t. each pair of indexes, i.e.

$$\mathcal{A}^{abcdef} = \mathcal{A}^{bacdef} = \mathcal{A}^{abdcef} = \mathcal{A}^{abcdfe} \; , \tag{A6}$$

quantity $\mathcal{B}_{abcdef}$ can be brought to a compact form

$$\mathcal{B}_{abcdef}(\mathbf{k_1}, \mathbf{k_2}, \mathbf{k_3}) = 8\delta(\mathbf{k_1} + \mathbf{k_2} + \mathbf{k_3})\int d^3\mathbf{q} M(|\mathbf{q}|)M(|\mathbf{k}_1 - \mathbf{q}|)M(|\mathbf{k}_2 + \mathbf{q}|)P_{ac}(\mathbf{q})P_{be}(\mathbf{k}_1 - \mathbf{q})P_{df}(\mathbf{k}_2 + \mathbf{q}) \; . \tag{A7}$$

3. Useful relations

Following relation from vector algebra is valid (see e.g. Eq. (31), p.16 [29]):

$$[\mathbf{A}\times\mathbf{B}]\cdot[\mathbf{C}\times\mathbf{D}] = (\mathbf{A}\cdot\mathbf{C})\ (\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})\ (\mathbf{B}\cdot\mathbf{C})\ . \tag{A8}$$

Using plane wave decomposition into spherical harmonics one obtains following useful representation for the Dirac delta function

$$\delta(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}) = \int \frac{d^3x}{(2\pi)^3}\mathrm{e}^{i(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})\mathbf{x}} = 8\int d^3x \sum_{t_1r_1}\sum_{t_2r_2}\sum_{t_3r_3} i^{t_1+t_2+t_3} j_{t_1}(k_1x)j_{t_2}(k_2x)j_{t_3}(k_3x)$$
$$Y_{t_1r_1}(\hat{\mathbf{k}}_1)Y^\star_{t_1r_1}(\hat{\mathbf{x}})Y_{t_2r_2}(\hat{\mathbf{k}}_2)Y^\star_{t_2r_2}(\hat{\mathbf{x}})Y_{t_3r_3}(\hat{\mathbf{k}}_3)Y^\star_{t_3r_3}(\hat{\mathbf{x}})\ . \tag{A9}$$

[1] N. Bartolo, S. Matarrese and A. Riotto, "Non-Gaussianity and the Cosmic Microwave Background Anisotropies," Adv. Astron. **2010**, 157079 (2010) [arXiv:1001.3957 [astro-ph.CO]].

[2] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, "Theory of cosmological perturbations. Part 1. Classical perturbations. Part 2. Quantum theory of perturbations. Part 3. Extensions," Phys. Rept. **215**, 203 (1992).

[3] E. Komatsu *et al.* [WMAP Collaboration], "Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation," Astrophys. J. Suppl. **192**, 18 (2011) [arXiv:1001.4538 [astro-ph.CO]].

[4] F. Bernardeau and J. P. Uzan, "Inflationary models inducing non-Gaussian metric fluctuations," Phys. Rev. D **67**, 121301 (2003) [arXiv:astro-ph/0209330].

[5] E. Komatsu, "Hunting for Primordial Non-Gaussianity in the Cosmic Microwave Background," Class. Quant. Grav. **27**, 124010 (2010).

[6] T. R. Seshadri and K. Subramanian, "CMB bispectrum from primordial magnetic fields on large angular scales," Phys. Rev. Lett. **103**, 081303 (2009) [arXiv:0902.4066 [astro-ph.CO]].

[7] C. Caprini, F. Finelli, D. Paoletti and A. Riotto, "The cosmic microwave background temperature bispectrum from scalar perturbations induced by primordial magnetic fields," JCAP **0906**, 021 (2009)

[8] I. Brown and R. Crittenden, "Non-Gaussianity from Cosmic Magnetic Fields," Phys. Rev. D **72**, 063002 (2005) [arXiv:astro-ph/0506570]; I. A. Brown, "Intrinsic Bispectra of Cosmic Magnetic Fields," Astrophys. J. **733**, 83 (2011) [arXiv:1012.2892 [astro-ph.CO]].

[9] P. Trivedi, K. Subramanian and T. R. Seshadri, "Primordial Magnetic Field Limits from Cosmic Microwave Background Bispectrum of Magnetic Passive Scalar Modes," Phys. Rev. D **82**, 123006 (2010) [arXiv:1009.2724 [astro-ph.CO]].

[10] K. Subramanian and J. D. Barrow, "Magnetohydrodynamics in the early universe and the damping of noninear Alfven waves," Phys. Rev. D **58**, 083502 (1998) [arXiv:astro-ph/9712083]; K. Subramanian and J. D. Barrow, "Microwave Background Signals from Tangled Magnetic Fields," Phys. Rev. Lett. **81**, 3575 (1998) [arXiv:astro-ph/9803261].

[11] A. Mack, T. Kahniashvili and A. Kosowsky, "Microwave background signatures of a primordial stochastic magnetic field," Phys. Rev. D **65**, 123004 (2002).

[12] A. Lewis, "CMB anisotropies from primordial inhomogeneous magnetic fields," Phys. Rev. D **70**, 043011 (2004)

[13] L. M. Widrow, "Origin of Galactic and Extragalactic Magnetic Fields," Rev. Mod. Phys. **74**, 775 (2002); A. Kandus, K. E. Kunze and C. G. Tsagas, "Primordial magnetogenesis," Phys. Rept. **505**, 1 (2011) [arXiv:1007.3891 [astro-ph.CO]].

[14] R. Durrer and C. Caprini, "Primordial Magnetic Fields and Causality," JCAP **0311**, 010 (2003).

[15] M. S. Turner and L. M. Widrow, "Inflation Produced, Large Scale Magnetic Fields," Phys. Rev. D **37**, 2743 (1988); B. Ratra, "Cosmological 'seed' magnetic field from inflation," Astrophys. J. **391**, L1 (1992); K. Bamba and M. Sasaki, "Large-scale magnetic fields in the inflationary universe," JCAP **0702**, 030 (2007).

[16] D. G. Yamazaki, K. Ichiki, T. Kajino and G. J. Mathews, "New Constraints on the Primordial Magnetic Field," Phys. Rev. D **81**, 023008 (2010); D. G. Yamazaki, K. Ichiki, T. Kajino and G. J. Mathews, "Constraints on the neutrino mass and the primordial magnetic field from the matter density fluctuation parameter σ_8," Phys. Rev. D **81**, 103519 (2010); D. Paoletti and F. Finelli, "CMB Constraints on a Stochastic Background of Primordial Magnetic Fields," Phys. Rev. D **83**, 123533 (2011) [arXiv:1005.0148 [astro-ph.CO]].

[17] A. Neronov and I. Vovk, "Evidence for strong extragalactic magnetic fields from Fermi observations of TeV blazars," Science **328**, 73 (2010); F. Tavecchio, et al., F. Tavecchio, G. Ghisellini, L. Foschini, G. Bonnoli, G. Ghirlanda and P. Coppi, 'The intergalactic magnetic field constrained by Fermi/LAT observations of the TeV blazar 1ES 0229+200," Mon. Not. Roy. Astron. Soc. **406**, L70 (2010) arXiv:1004.1329 [astro-ph.CO]; F. Tavecchio, G. Ghisellini, G. Bonnoli and L. Foschini, "Extreme TeV blazars and the intergalactic magnetic field," arXiv:1009.1048 [astro-ph.HE];
A. M. Taylor, I. Vovk and A. Neronov, "Extragalactic magnetic fields constraints from simultaneous GeV-TeV observations of blazars," Astron. Astrophys. **529**, A144 (2011) [arXiv:1101.0932 [astro-ph.HE]]; K. Takahashi, M. Mori, K. Ichiki and S. Inoue, "Lower Bounds on Intergalactic Magnetic Fields from Simultaneously Observed GeV-TeV Light Curves of the Blazar Mrk 501," arXiv:1103.3835 [astro-ph.CO]; C. D. Dermer, M. Cavadini, S. Razzaque, J. D. Finke and B. Lott, "Time

Delay of Cascade Radiation for TeV Blazars and the Measurement of the Intergalactic Magnetic Field," arXiv:1011.6660 [astro-ph.HE].

[18] K. Dolag, M. Kachelriess, S. Ostapchenko and R. Tomas, "Lower limit on the strength and filling factor of extragalactic magnetic fields," Astrophys. J. **727**, L4 (2011) [arXiv:1009.1782 [astro-ph.HE]];

[19] M. Shiraishi, S. Yokoyama, K. Ichiki and K. Takahashi, "Analytic formulae of the CMB bispectra generated from non-Gaussianity in the tensor and vector perturbations," Phys. Rev. D **82**, 103505 (2010) [arXiv:1003.2096 [astro-ph.CO]].

[20] M. Shiraishi, D. Nitta, S. Yokoyama, K. Ichiki and K. Takahashi, "Cosmic microwave background bispectrum of vector modes induced from primordial magnetic fields," Phys. Rev. D **82**, 121302 (2010) [Erratum-ibid. D **83**, 029901 (2011)] [arXiv:1009.3632 [astro-ph.CO]].

[21] M. Shiraishi, D. Nitta, S. Yokoyama, K. Ichiki and K. Takahashi, "Computation approach for CMB bispectrum from primordial magnetic fields," Phys. Rev. D **83**, 123523 (2011) [arXiv:1101.5287 [astro-ph.CO]].

[22] M. Shiraishi, D. Nitta, S. Yokoyama, K. Ichiki and K. Takahashi, "Cosmic microwave background bispectrum of tensor passive modes induced from primordial magnetic fields," Phys. Rev. D **83**, 123003 (2011) [arXiv:1103.4103 [astro-ph.CO]].

[23] J. M. Bardeen, "Gauge Invariant Cosmological Perturbations," Phys. Rev. D **22**, 1882 (1980); R. Durrer, "Gauge Invariant Cosmological Perturbation Theory: A General Study And Its Application To The Texture Scenario Of Structure Formation," Fund. Cosmic Phys. **15**, 209 (1994).

[24] W. Hu and M. J. White, "CMB Anisotropies: Total Angular Momentum Method," Phys. Rev. D **56**, 596 (1997). [arXiv:astro-ph/9702170].

[25] A. Brandenburg, K. Enqvist and P. Olesen, "Large-scale magnetic fields from hydromagnetic turbulence in the very early universe," Phys. Rev. D **54**, 1291 (1996) [arXiv:astro-ph/9602031].

[26] K. Jedamzik, V. Katalinic and A. V. Olinto, "Damping of cosmic magnetic fields," Phys. Rev. D **57**, 3264 (1998) [astro-ph/9606080].

[27] R. Durrer, T. Kahniashvili and A. Yates, "Microwave Background Anisotropies from Alfven waves," Phys. Rev. D **58**, 123004 (1998) [arXiv:astro-ph/9807089].

[28] T. Kahniashvili, G. Lavrelashvili and B. Ratra, "CMB Temperature Anisotropy from Broken Spatial Isotropy due to an Homogeneous Cosmological Magnetic Field," Phys. Rev. D **78**, 063012 (2008).

[29] D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii, *Quantum Theory of Angular Momentum.* (World Scientific, Singapore, 1988).

[30] E. Komatsu and D. N. Spergel, "Acoustic signatures in the primary microwave background bispectrum," Phys. Rev. D **63**, 063002 (2001).

[31] I. Brown, 2009, private communication.